Section 11.1

Linear Approximations

We begin this section with a return to a topic from Chapter 4: Linear approximation of a differentiable function at a point $x = a$.

Example 11.1.1. Find the linear approximation of $p(x) = 7x^4 + 5x^3 - 3x^2 + 2x + 1$ at $x = 0$

Example 11.1.2. Find a polynomial with the same evaluation at $x = 0$ that also has the same first and second derivatives as $p(x)$. Continue the process so that the third derivatives coincide.

Definition 11.1.3. Taylor Polynomials

Assume f and its first n derivatives exist at $x = a$. The *nth-order Taylor polynomial* for f centered at a, denoted p_n, is the polynomial satisfying

$$p_n(a) = f(a), p_n'(a) = f'(a), \ldots, p_n^{(n)}(a) = f^{(n)}(a).$$

The n^{th}-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Written more compactly, $p_n(x) = \sum_{k=0}^{n} c_k(x-a)^k$, where

$$c_k = \frac{f^{(k)}(a)}{k!}.$$

Example 11.1.4. Find the first 4 Taylor polynomials for $f(x) = \sin x$ centered at $x = 0$.

Example 11.1.5. Find a formula for the coefficients c_k for a k¡nth-order Taylor polynomial.

Approximations using Taylor Polynomials

The approximation obtained from Taylor polynomials has very small error for relatively low values of n.

Example 11.1.6. Approximate $\sqrt{10}$ using a 3rd-order Taylor polynomial. Compute the absolute error.

Definition 11.1.7. Remainder in a Taylor Polynomial

Let p_n be the nth-order Taylor polynomial of f. the *remainder* in using p_n to approximate f at a point x is

$$R_n(x) = f(x) - p_n(x).$$

Theorem 11.1.8. Taylor's Theorem

Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I,

$$f(x) = p_n(x) + R_n(x),$$

where p_n is the nth-order Taylor polynomial of f centered at a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some point c between x and a.

Theorem 11.1.9. Bound on the Remainder

Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \leq M$, for all c between a and x inclusive. Then the remainder in the nth-order Taylor polynomial of f centered at a satisfies

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

Example 11.1.10. Estimate the remainder in approximating $e^{0.45}$ using the 6th-order Taylor polynomial.